Ch. 4: Techniques of Circuit Analysis

So far we have analyzed simple resistive circuits using Ohm's Law and Kirchhoff current and voltage laws. These were also used to derive voltage and current division rules and rules to combine resistances in series and parallel, as well as Δ-Y transformations for circuit simplification.

In this chapter we use KCL and KVL to develop two circuit analysis techniques that will allow us to tackle more complex circuits: these are the node-voltage method and mesh-current method.

We also introduce the Thevenin and Norton equivalent circuits and source transformation.

4.1 Terminology

A planar circuit is one that can be drawn on a paper (plane) without any crossing branches. The circuit in Fig. 4.1 (a) is planar, whereas the circuit in Fig. 4.2 is not planar.

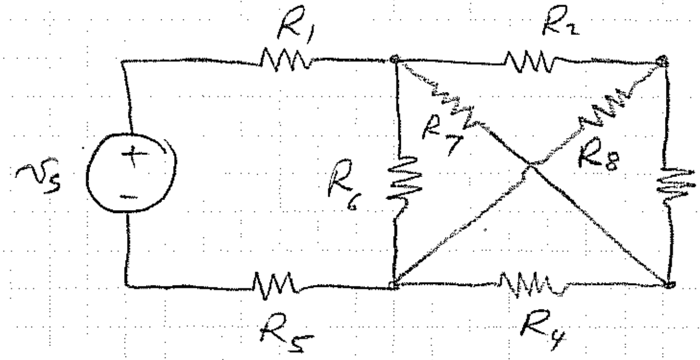


Fig. 4.1(a) Planar Circuit.

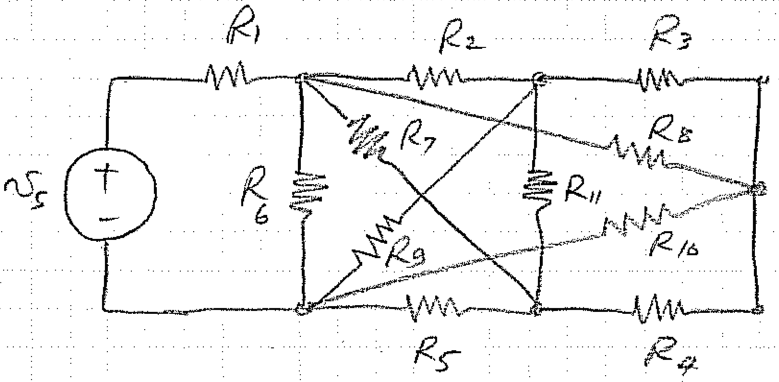


Fig. 4.1 (b) Non-planar Circuit.

The circuit in 4.1(b) can be redrawn with the same branch-node connections and no branch overlap!

The node-voltage method is applicable to all circuits planar and non-planar, whereas the mesh-current method is applicable to planar circuits only.

The vocabulary used describing a circuit is illustrated in the following example:

Example 4.1: Identify nodes, branches, paths meshes and loops in a circuit.

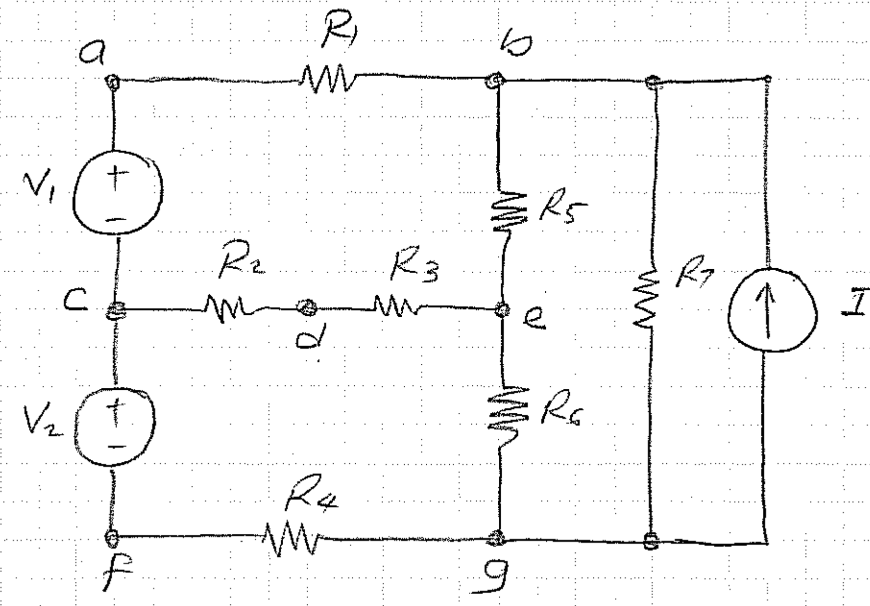


Fig. 4.3

a) The nodes are a, b, c, d, e, f, and g

b) The essential nodes are b, c, e, and g.

c) The branches are V_1, V_2, R_1, R_2, R_3, R_4, R_5, R_6, R_7, and I.

d) The essential branches are V_1-R_1, V_2-R_2, R_2-R_3, R_5, R_6, R_7 and I

e) The meshes are loops with no smaller loops inside them: V_1-R_1-R_5-R_3-R_2, V_2-R_2-R_3-R_6-R_4, R_5-R_7-R_6, and R_7-I.

f) V_1-R_1-R_5-R_6-R_4-V_2 is a loop, but not a mesh since it imbeds two loops within it. I-R_5-R_6 is also a loop but not a mesh.

g) A path connects several elements but is not a loop nor an essential branch: $R_1 - R_5 - R_6$ is a path, $V_2 - R_2$ is also a path.

Simultaneous Equations: How Many?

The number of unknown currents is equal to the number of essential branches B_e, which is equal to 7 in the circuit of Fig. 4.3. Once these unknown currents are determined all power generation and dissipation in the circuit can be calculated.

The number of essential nodes (N_e) in the above circuit is 4. When the voltages at $N_e - 1$ nodes are known with respect to one of them, then the currents in all branches can be determined. In the above circuit the number of node voltage equations needed is $N_e - 1 = 3$.

Without proof, the number of meshes is equal to $M = B_e - (N_e - 1)$, which is equal to 4 in the above circuit. So 4 mesh-current equations describe the steady-state operation of the above circuit.

4.2 The Node-Voltage Method.

The method is illustrated with the circuit of Fig. 4.5, which has 3 essential nodes. We select one of the nodes to be a reference node and mark it with the symbol ▼ and mark the other two with their respective numbers.

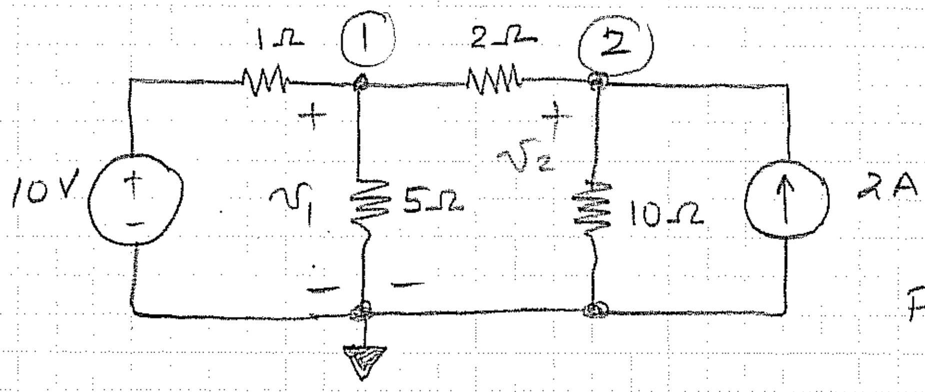


Fig. 4.5

Then we define the node voltages at nodes 1, and 2 with respect to the reference

The node-voltage equations are essentially KCL using the defined node-voltages with ohm's law imbedded in them.

KCL at node 1:

$$\frac{v_1 - 10}{1} + \frac{v_1}{5} + \frac{v_1 - v_2}{2} = 0 \qquad (4.5)$$

KCL at node 2:

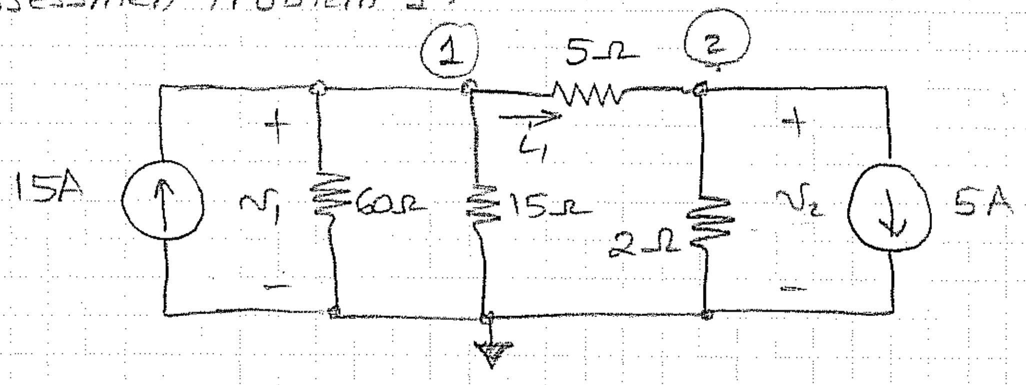
$$\frac{v_2 - v_1}{2} + \frac{v_2}{10} - 2 = 0 \qquad (4.6)$$

Solving for v_1 and v_2 we obtain:

$$v_1 = \frac{100}{11} = 9.09V$$

$$v_2 = \frac{120}{11} = 10.91V$$

Assessmen Problem 1:



For the above circuit:

a) Find v_1, v_2, and i_1

b) Power delivered by the 15A source

c) Power delivered by the 5A source.

Solution:

a) The circuit has 3 essential nodes. The bottom node is selected as reference and the other two are marked 1 and 2.

Once could have selected node 1 to be the reference node and name the other two one and two. But the selected naming above makes the nodal voltages v_1 and v_2 coincide with the requested unknowns.

KCL at node 1:

$$-15 + \frac{v_1}{60} + \frac{v_1}{15} + \frac{v_1 - v_2}{5} = 0 \qquad (1)$$

KCL at node 2:

$$\frac{v_2 - v_1}{5} + \frac{v_2}{2} + 5 = 0 \qquad (2)$$

Rearrange equations:

$$\left(\frac{1}{60} + \frac{1}{15} + \frac{1}{5}\right) v_1 - \frac{1}{5} v_2 = 15 \qquad (1a)$$

$$-\frac{1}{5} v_1 + \left(\frac{1}{5} + \frac{1}{2}\right) v_2 = -5 \qquad (2a)$$

which give:

$$0.2833 v_1 - 0.2 v_2 = 15 \quad (\times 0.7) \quad (1a)$$

$$-0.2 v_1 + 0.7 v_2 = -5 \quad (\times 0.2) \quad (1b)$$

Multiply first equation by 0.7 and the second by 0.2 and Add.

$$0.19831 v_1 - 0.04 v_1 = 10.5 - 1$$

So $v_1 = \dfrac{9.5}{0.15831} = 60\text{ V}$

Replace in (1b) and get v_2:

$$v_2 = \frac{12 - 5}{0.7} = 10\text{V}$$

$$i_1 = \frac{v_1 - v_2}{5} = \frac{60-10}{5} = 10\,A$$

b) $P_{15A} = 60 \times 15 = 900\,W$

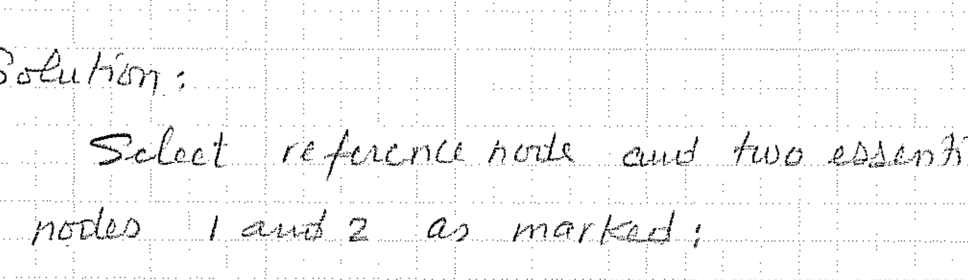
c) $P_{5A} = -5 \times 10 = -50\,W$

The 5A source is actually consuming 50W. It is being charged "so to speak".

4.3 The Node-Voltage Method and Dependent Sources

Example 4.3

Use the node-voltage method to find the power dissipated in the 5Ω resistor in the circuit shown below.



Solution:

Select reference node and two essential nodes 1 and 2 as marked:

KCL at node 1: $\frac{v_1 - 20}{2} + \frac{v_1}{20} + \frac{v_1 - v_2}{5} = 0$

KCL at node 2: $\frac{v_2 - v_1}{5} + \frac{v_2}{10} + \frac{v_2 - 8i_\phi}{2} = 0$

But $i_\phi = \frac{v_1 - v_2}{5}$

Replace in the second equation above and rearrange:

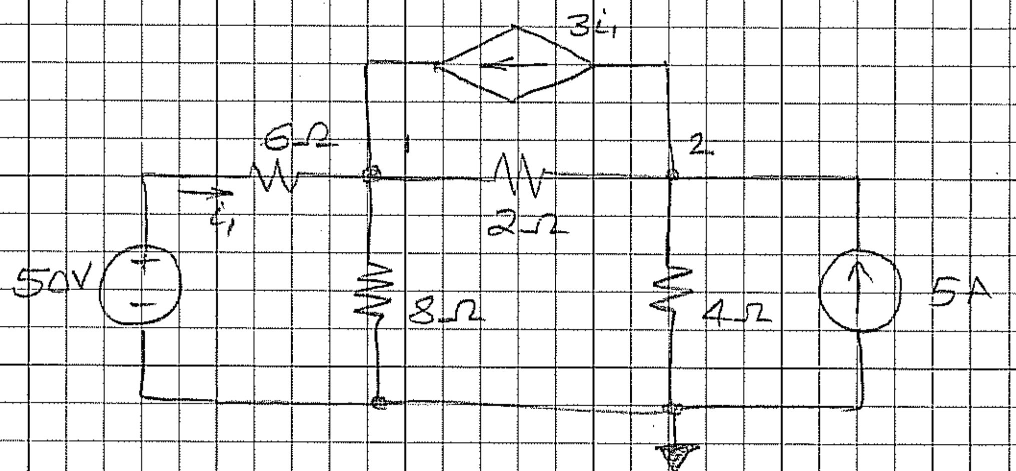
$$0.75 v_1 - 0.2 v_2 = 10 \qquad (1)$$

$$-v_1 + 1.6 v_2 = 0 \qquad (2) \qquad \times 0.75$$

We obtain: $v_2 = 10\,V$ and $v_1 = 16\,V$.

Assessment Problem:

a) Use the node-voltage to find the power delivered by each source in the circuit shown.



KCL at node 1: $\frac{v_1 - 50}{6} + \frac{v_1}{8} + \frac{v_1 - v_2}{2} - 3i_1 = 0$

KCL at node 2: $\frac{v_2 - v_1}{2} + 3i_1 + \frac{v_2}{4} - 5 = 0$

But the current $i_1 = \frac{50 - v_1}{6}$

Replace in above equations and rearrange:

$$1.291667\, v_1 - 0.5\, v_2 = 33.33$$

$$-v_1 + 0.75\, v_2 = -20 \qquad \times 0.6667$$

$$0.625\, v_1 + 0 = 20$$

$$v_1 = 32\ \text{V}$$

$$v_2 = \frac{32-20}{0.75} = 16\,\text{V}$$

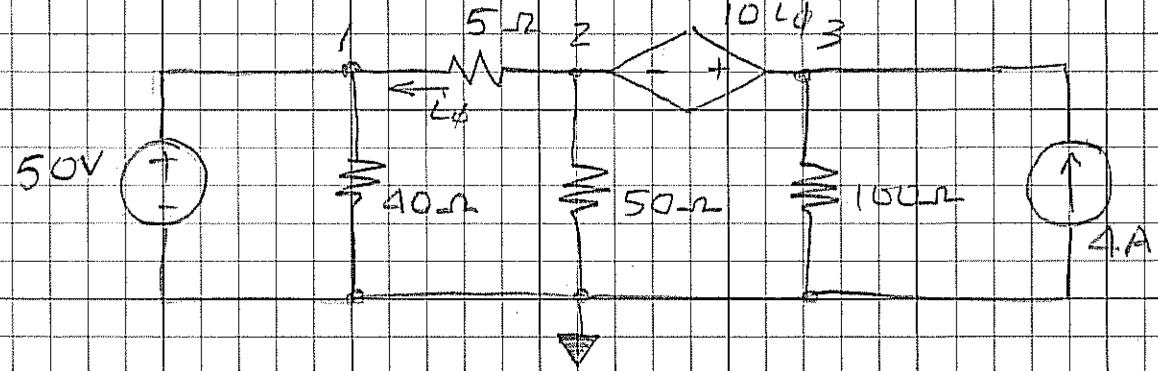
$P_{5A} = 16 \times 5 = 80\,\text{W}$ delivered to network.

$P_{3i} = 3i_1 \times (v_1 - v_2) = 3 \times \left(\frac{50-32}{6}\right)(32-16)$
$= 144\,\text{W}$ delivered to network.

$P_{50V} = 50\, i_1 = 50 \times \left(\frac{50-32}{6}\right) = 150\,\text{W}$ also delivered to the network!

4.4 Some Special Cases

When a voltage source is the only element between two essential nodes, the node voltage method is simplified, as illustrated in the following example:



There are 4 essential nodes, where one (bottom) is selected as reference and the other 3 are as marked: 1, 2, and 3.

The voltage v_1 at node 1 is 50V, and hence there is no need to write KCL at node 1:

$$v_1 = 50\,V \qquad (1)$$

A node 2 it will be difficult to write KCL because of the voltage source, similary it will be difficult to write KCL at node 3. However we know that $\sum_{node 2} i = 0$ and $\sum_{node 3} i = 0$, therefore the sum of current at "Supernode" 2-3 is also equal to zero:

$$\frac{v_2 - 50}{5} + \frac{v_2}{50} - 4 + \frac{v_3}{100} = 0 \qquad (2) ✓$$

The voltage source between nodes 2 and 3 relates v_2 and v_3 as follows:

$$v_3 - v_2 = 10\, i_\phi \qquad (\text{KVL})$$

but $i_\phi = \dfrac{v_2 - 50}{5}$

So $\quad v_3 - v_2 = \left(\dfrac{v_2 - 50}{5}\right) 10$

Rearrange to obtain:

$$3v_2 - v_3 = 100 \qquad (3)$$

Rearrange (2):

$$0.22 v_2 + 0.01 v_3 = 14 \qquad (2a)$$

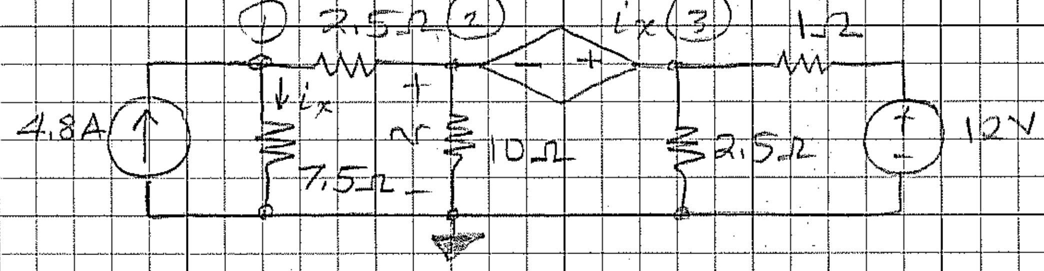
Multiply (3) by 0.01 and add to (2a):

$$0.25 v_2 = 15 \Rightarrow v_2 = 60\,V$$

From (3) $\Rightarrow$ $v_3 = 3\times 60 - 100 = 80\,V$

Assessment Problem 4.5

Use the node-voltage method to find v in the following circuit



The circuit has 4 essential nodes, the bottom one is selected as a reference and the top 3 are marked 1, 2, and 3 as shown.

The network equations are developed as follows:

KCL at (1): $-4.8 + \dfrac{v_1}{7.5} + \dfrac{v_1 - v_2}{2.5} = 0 \qquad (1)$

KCL at supernode 2-3, containing voltage source:

$$\frac{v_2 - v_1}{2.5} + \frac{v_2}{10} + \frac{v_3}{2.5} + \frac{v_3 - 12}{1} = 0 \qquad (2)$$

The last equation is that of the dependent voltage source relating v_2 and v_3:

$$v_3 - v_2 = i_x = \frac{v_1}{7.5} \qquad (3)$$

Solve the 3 equations (1), (2), and (3) for v_1, v_2 and v_3!

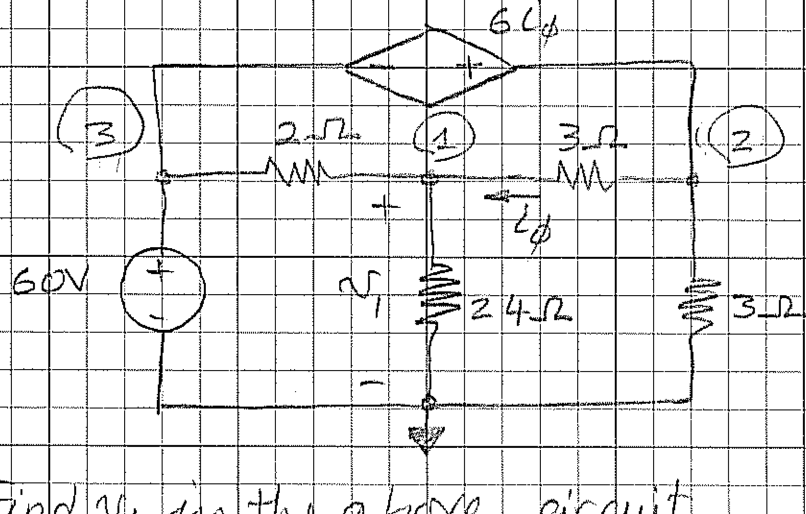
$$(1) \Rightarrow \quad 0.5333\, v_1 - 0.4\, v_2 + 0\, v_3 = 4.8 \qquad (1a)$$

$$(2) \Rightarrow \quad -0.4\, v_1 + 0.5\, v_2 + 1.4\, v_3 = 12 \qquad (2a)$$

$$(3) \Rightarrow \quad -0.133\, v_1 - v_2 + v_3 = 0$$

$$v_1 = 15\,V, \quad v_2 = 8\,V, \quad v_3 = 10\,V.$$

Assessment Problem 4.6



Find v_1 in the above circuit

The above circuit has 4 essential nodes. The bottom one is selected as reference. The others are marked as 1, 2, and 3 as shown.

The voltage at node 3 is known, $v_3 = 60V$!
We need therefore two equations in two unknowns v_1 and v_2.

KCL at node 1 provides the first equation:

$$\frac{v_1 - 60}{2} + \frac{v_1}{24} + \frac{v_1 - v_2}{3} = 0$$

Rearrange:

$$0.875 v_1 - 0.333 v_2 = 30 \qquad (1)$$

The second equation is that of the dependent voltage source:

$$v_2 - 60 = 6 i_\phi = 6 \times \left(\frac{v_2 - v_1}{3} \right)$$

Rearrange

$$2v_1 - v_2 = 60 \qquad (2)$$

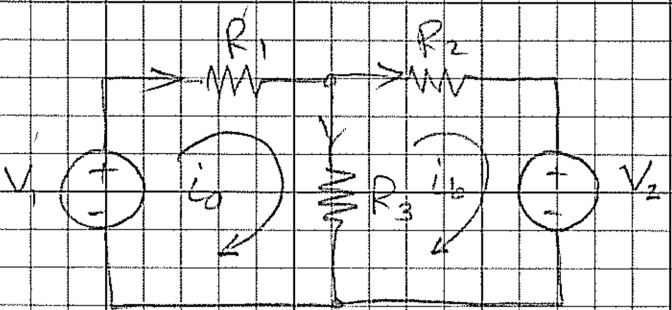
The solution is: $v_1 = 48V$ $v_2 = 36V$

Note that in this problem there was no need to use the supernode. The supernode is to be used when the voltage is connecting two essential nodes with unknown voltages.
in Assess Problem 4.4 no need to use a supernode

4.5 Introduction of the Mesh-Current Method.

Recall that a mesh is a loop with no other loops imbedded in it. The mesh-current method enables us to describe a circuit in terms of $B_e - (N_e - 1)$ equations.

Consider the circuit shown below, with 2 mesh currents defined, i_a and i_b.



A mesh-current exists only in the perimeter of a mesh. In the above circuit the current in R_1 is equal to i_a and the current in R_2 is equal to i_b, and the current in R_3 is $i_a - i_b$.

The number of mesh-current equations in the above circuit is $2 (= 3 - (2-1))$, corresponding for meshes a and b.

The mesh current equation are KVL equations written in terms of the mesh-currents:

KVL for Mesh a:

$$-V_1 + i_a R_1 + (i_a - i_b) R_3 = 0 \qquad (1)$$

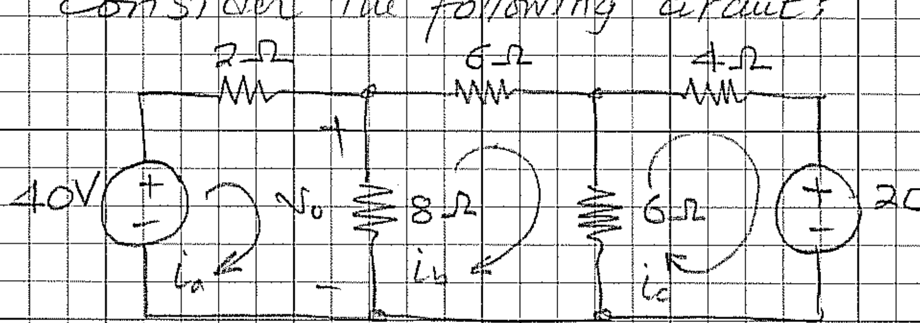
KVL for Mesh b:

$$(i_b - i_a) R_3 + i_b R_2 + V_2 = 0 \qquad (2)$$

The two unknowns in equations (1) and (2) are i_a and i_b.

Example 4.4

Consider the following circuit:



a) Use the MCM to determine the power associated with each voltage source.

b) Calculate v_o across the 8Ω resistor

Solution:

We need 3 mesh-current equations. $B_e = 5$, $N_e = 3$

So the number of meshes is: $M_e = 5-(3-1) = 3$!

Define the mesh currents i_a, i_b, and i_c, and write the KVL equations:

$$-40 + 2i_a + 8(i_a - i_b) = 0 \qquad (1)$$

$$8(i_b - i_a) + 6i_b + 6(i_b - i_c) = 0 \qquad (2)$$

$$6(i_c - i_b) + 4i_c + 20 = 0 \qquad (3)$$

Rearrange the equations:

$$10i_a - 8i_b + 0i_c = 40 \qquad (1a)$$

$$-8i_a + 20i_b - 6i_c = 0 \qquad (2a)$$

$$0i_a - 6i_b + 10i_c = -20 \qquad (3a)$$

Solve using calculator:

$$i_a = 5.6\,A, \quad i_b = 2\,A, \quad i_c = -0.8\,A$$

Power associated with Sources:

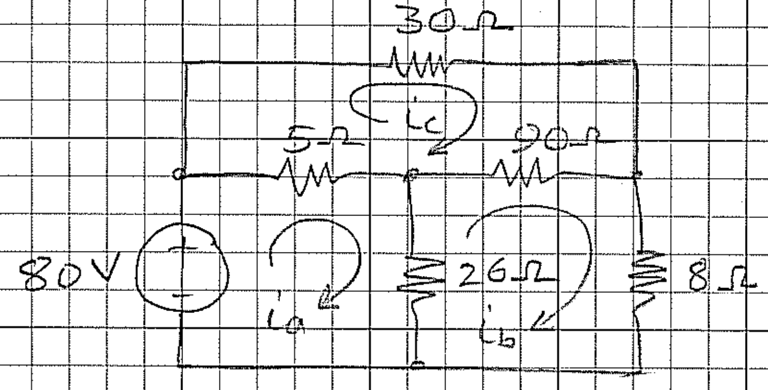
$$P_{40V} = -40i_a = -224\,W$$

$$P_{20V} = 20i_c = 20 \times (-0.8) = -16\,W$$

Both sources are delivering power to the network.

Assessment Problem 4.7

Use the MCM to find a) the power delivered by the 80V source and b) the power dissipated in the 8 Ω resistor:



The number of meshes is 3 = 6 − (4−1). So define the three mesh currents i_a, i_b and i_c

The mesh-current equations are:

$$-80 + 5(i_a - i_c) + 26(i_a - i_b) = 0 \qquad (1)$$
$$26(i_b - i_a) + 90(i_b - i_c) + 8i_b = 0 \qquad (2)$$
$$5(i_c - i_a) + 30i_c + 90(i_c - i_b) = 0 \qquad (3)$$

Rearrange and Solve:

$$31i_a - 26i_b - 5i_c = 80$$
$$-26i_a + 124i_b - 90i_c = 0$$
$$-5i_a - 90i_b + 125i_c = 0$$

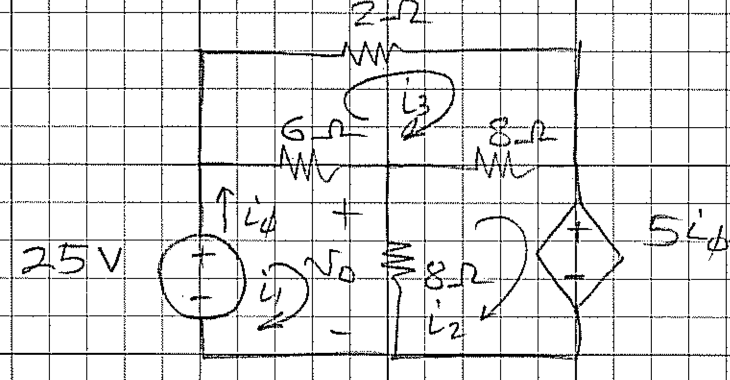
$i_a = 5A$, $i_b = 2.5A$, $i_c = 2$

a) $P_{80V} = 80 \times 5 = 400W$

b) $P_{8\Omega} = 8 \times (2.5)^2 = 50W$

4.6 The Mesh-Current Method and Dependent Sources

Assessment Problem 4.9



Use the mesh-current method to find v_0.

$N_e = 4$, $B_e = 6 \Rightarrow M_e = 6-(4-1) = 3$

Define the mesh currents i_1, i_2, and i_3 and write the 3 KVL equations:

$$-25 + 6(i_1 - i_3) + 8(i_1 - i_2) = 0 \qquad (1)$$

$$8(i_2 - i_1) + 8(i_2 - i_3) + 5i_\phi = 0 \qquad (2)$$

but $i_\phi = i_1$!

The 3rd equations

$$6(i_3 - i_1) + 2i_3 + 8(i_3 - i_2) = 0 \qquad (3)$$

Replace i_ϕ in (2) by i_1 and rearrange all equations:

$$14i_1 - 8i_2 - 6i_3 = 25 \qquad (1a)$$

$$-3i_1 + 16i_2 - 8i_3 = 0 \qquad (1b)$$

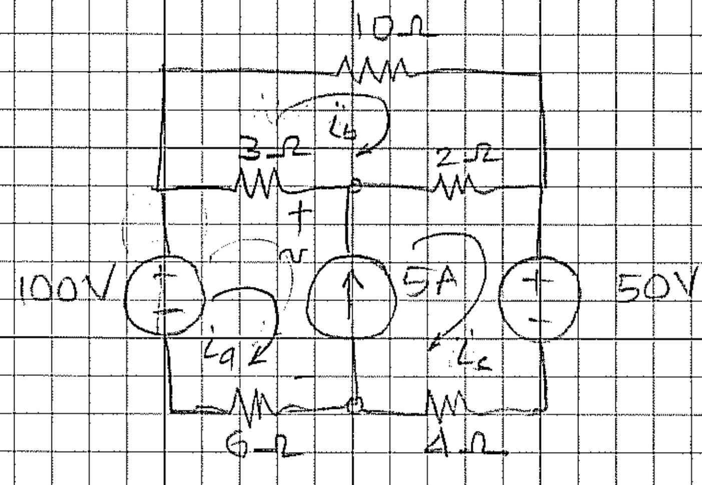
$$-6i_1 - 8i_2 + 16i_3 = 0 \qquad (1c)$$

Solve: $i_1 = 4,\ i_2 = 2,\ i_3 = 2.5\,A$

$$v_o = 8(i_1 - i_2) = 8(4-2) = 16\,V.$$

4.7 The Mesh-Current Method: Some Special Cases.

When a current source is in a branch between two meshes, then writing KVL for either is difficult as in the following examples



It will be difficult to write KVL for mesh a and mesh b in terms of the mesh current variables. So let the voltage across the current source be v.

KVL for mesh a:

$$-100 + 3(i_a - i_b) + v + 6i_a = 0 \qquad (1)$$

KVL for mesh c:

$$-v + 2(i_c - i_b) + 50 + 4i_c = 0 \qquad (2)$$

Add equations (1) and (2) and noting that the terms in v will cancel out:

$$-100 + 3(i_a - i_b) + 6i_a + 2(i_c - i_b) + 50 + 4i_c = 0 \quad (3)$$

Note that equation 3 is KVL writen for the "super-mesh" made from meshes a and c

This would be one equation to determine i_a, i_b and i_c. The second equation is KVL for mesh b:

$$10i_b + 2(i_b - i_c) + 3(i_b - i_a) = 0 \quad (4)$$

And the 3rd equation would be that of the current source:

$$i_c - i_a = 5 \quad (5)$$

Solve equations (3), (4) and (5) for i_a, i_b, and i_c. Rearrange them as:

$$9i_a - 5i_b + 6i_c = 50 \quad (3a)$$
$$-3i_a + 15i_b - 2i_c = 0 \quad (4a)$$
$$-i_a + 0i_b + i_c = 5 \quad (5a)$$

$$i_a = 1.75A, \quad i_b = 1.25A, \quad i_c = 6.75A$$